Modelling current-voltage characteristics of practical superconductors

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Outline

1. **Statement of the problem**
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   - The underlying physical problem
   - Macroscopic *material law*

2. **(Numerical) modelling**
   - Thermodynamic model: general framework
   - The power-law-like $\mathcal{F}(\mathbf{J})$ formulation
   - The power-law-like $\mathbf{E}(\mathbf{J})$ formulation

3. **Application**
   - Approximations to the *helical cable* geometry
   - Fingerprints of the $\mathbf{E}(\mathbf{J})$ law

4. **Conclusions**
1. Statement of the problem

1.1 Motivation

The Macroscopic Maxwell Equations must be supplied with a sound and practical expression of the superconducting material law.

In quasi-static conditions:

\[
\mathbf{E}(\mathbf{J}) = \rho(\mathbf{J}) \mathbf{J}
\]

\[
\downarrow
\]

\[
\left( \mu_0 \frac{\partial}{\partial t} - \rho(\mathbf{J}) \nabla^2 \right) \mathbf{H} = (\nabla \times \mathbf{H}) \times \nabla \rho(\mathbf{J})
\]

★ A number of particular choices exist for \( \rho(\mathbf{J}) \), but FE codes lack an implementation for general purpose.

★ \( \rho(\mathbf{J}) \) is not always a scalar, neither a tensor!!
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The Macroscopic Maxwell Equations must be supplied with a **SOUND** and **PRACTICAL** expression of the superconducting material law

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\begin{align*}
E(J) &= \rho(J)J \\
\downarrow \\
\left( \mu_0 \frac{\partial}{\partial t} - \rho(J) \nabla^2 \right) H &= (\nabla \times H) \times \nabla \rho(J)
\end{align*}
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★ A number of particular choices exist for \(\rho(J)\), but FE codes lack an implementation for general purpose

★ \(\rho(J)\) is not always a scalar, neither a tensor !!
1.2. The underlying physical problem

**LOCAL GEOMETRY OF AMPÈRE’S LAW** \((J\parallel, J\perp)\)

\[
1 \equiv \frac{H}{H} \quad ; \quad 2 \equiv \frac{\nabla H}{\|\nabla H\|} \quad ; \quad 3 \equiv 1 \times 2
\]

\[
\Rightarrow \quad J = H(-\partial_2 \theta + \partial_3 \phi)1 + (H\partial_1 \theta)2 + (H\partial_1 \phi - \partial_2 H)3
\]

**EXAMPLE 1:** uniform current density + axial field

\[
1 = (-y, x, 1)/\sqrt{1 + \rho^2}
\]
\[
2 = (x, y, 0)/\rho
\]
\[
3 = (-y, x, -\rho^2)/\rho \sqrt{1 + \rho^2}
\]

\[
J_1 = J_0/\sqrt{1 + \rho^2} = -H\partial_2 \theta
\]

\[
J_2 = 0
\]

\[
J_3 = -J_0 \rho \sqrt{1 + \rho^2} = -\partial_2 H
\]
1.3. The underlying physical problem

Local Geometry of Ampère’s law \((J_\parallel, J_\perp)\)

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**EXAMPLE 2:** planar sample in rotating field

\[
\begin{align*}
J_\parallel & \text{ only comes from the tilt between adjacent layers } (-\partial_2 \theta) 
\end{align*}
\]
The appearance of $J_\parallel$ ...
THE DISAPPEARANCE OF $J_{\parallel} \ldots$
Partial conclusions

A) ★ Rotations of the magnetic field are shielded by $J_\parallel$
B) ★ In MQS, when rotation ceases $J_\parallel$ disappears

Here, we have solved: $\nabla^2 H = (\mu_0/\rho_0) \frac{\partial H}{\partial t}$

then

$J \cdot H = 0 \Rightarrow \frac{\partial (H_x/H_y)}{\partial t} = 0$

In a superconductor
A) is true
B) both $J_{\parallel}$ and $J_{\perp}$ persist in MQS regime
Partial conclusions

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$$\nabla^2 \mathbf{H} = \left( \frac{\mu_0}{\rho_0} \right) \frac{\partial \mathbf{H}}{\partial t}$$

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In a superconductor

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1.3. Material law in type-II superconductors

★ Electromagnetic energy of the Vortex Lattice

\[ W_{SC} = \frac{1}{\mu_0} \int_{\Omega} \mathbf{V} \cdot \left( \mathbf{b}_1 + \frac{1}{2} \mathbf{b}_2 - \mu_0 \mathbf{H} \right) \]

\[ \mathbf{V} = \sum_i \Phi_0 \delta^2(\mathbf{r} - \mathbf{r}_i) \mathbf{n}_i: \text{ vorticity} \]

\[ \mathbf{b}_2 \text{ flux density of the equilibrium Vortex Lattice} \]

\[ \mathbf{b}_1 \text{ flux related to other sources} \]

\[ \mathbf{H} \text{ field intensity: } \nabla \times \mathbf{H} = \mathbf{J}_0 \]

★ The equilibrium (\( \partial_\eta W_{SC} = 0 \)) is given by a triangular vortex lattice with a uniform macroscopic field \( \mathbf{B} \) parallel to \( \mathbf{H} \). Then \( \mathbf{B} = \mu_0 \mathbf{H} \) and \( \nabla \times \mathbf{B} = 0 \) well within the sample.
In non-ideal (practical) superconductors, \( B \) may vary in intensity (\( J_\perp \)) and orientation (\( J_\parallel \)).

Then: \( W_{\text{Full}} = W_{\text{SC}} + W_{\text{Pinning}} \)

Equilibrium: \( \partial_\eta W_{\text{SC}} + \partial_\eta W_{\text{Pinning}} \) (forces + constraints = 0)

\[
J_\perp \propto F_p^{\perp} = F_p \cos \alpha; \quad J_\parallel \propto \tau_p \propto F_p^{\parallel} = F_p \sin \alpha \Rightarrow \frac{J_\perp^2}{a^2} + \frac{J_\parallel^2}{b^2} = 1
\]
Clarifying E(J): CWDC experiment


An elliptic $J_\parallel(J_\perp)$ law has been reported
Clarifying $E(J)$: CWDC experiment


Eq. (25) corresponds to the Critical State Theory ...

that postulates a non-functional relation \( \{E, J\} \Rightarrow J \in \Delta \)
2. Numerical Modelling  \( \rightarrow \) \( E(J) \)

2.1. Thermodynamic model (SST 2012)

Minimize \( C \equiv \frac{\mu_0}{2} \int_{\mathbb{R}^3} \| H_{n+1} - H_n \|^2 + \Delta t \int_{\Omega} \mathcal{F}[J] \)

\[
E = \nabla J \mathcal{F}
\]
Academic 1D example: transport along type-II cylinder with quasi-linear $E(J)$
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Towards 3D modelling: expanding the yield region

SST 2012, Badía & López
Expanded yield region: first example

Transport along crossed tapes
**Expanded yield region: first example**

Transport along crossed tapes
2.2. The power-law-like $\mathcal{F}(J)$ formulation

In SST 2012 it was shown that

$$\mathcal{F}_{\text{QLL}}(J) = \frac{1}{2} \rho \Theta(T)(J \pm J_{c\perp})^2 \quad \text{(Quasi-linear-law)}$$

$\&$

$$\mathcal{F}_{\text{PL}}(J) = F_0 \left( \frac{J}{J_{c\perp}} \right)^M, \quad M \gg 1 \quad \text{(Power-law)}$$

are equivalent in 1D

$\star$ Here, we generalize $\mathcal{F}_{\text{PL}}$ to 3D

$$\mathcal{F}_{\text{PL}}(\mathbf{J}) = F_0 \left[ \left( \frac{J_{\parallel}}{J_{c\parallel}} \right)^2 + \left( \frac{J_{\perp}}{J_{c\perp}} \right)^2 \right]^M$$
2.3. The power-law-like $E(J)$ formulation

$$E_{PL}(J) = \nabla_J \mathcal{F}_{PL}(J)$$

\[\downarrow\]

$$e(j) = \left( j^2 + \gamma j^2 \right)^{M-1} \left( j + \gamma j_\parallel \right)$$

with the definitions:

$$\gamma \equiv J^2_{c\perp}/J^2_{c\parallel} - 1 \equiv \Gamma^2 - 1$$

$$j \equiv J/J_{c\perp}$$

$$e \equiv E/(2MF_0J_{c\perp})$$

* Applied to CWDC experiment:

$$\frac{e_y}{e_z} = \frac{\gamma \sin \alpha \cos \alpha}{1 + \gamma \cos^2 \alpha} = \frac{(\Gamma^2 - 1) \tan \alpha}{\Gamma^2 + \tan^2 \alpha}$$
2.3. The power-law-like $E(J)$ formulation

$$E_{PL}(J) = \nabla_J F_{PL}(J)$$

$\downarrow$

$$e(j) = \left(j^2 + \gamma j_\parallel^2\right)^{M-1} \left(j + \gamma j_\parallel\right)$$

with the definitions:

$$\gamma \equiv J_{c\perp}^2 / J_{c\parallel}^2 - 1 \equiv \Gamma^2 - 1$$

$$j \equiv J / J_{c\perp}$$

$$e \equiv E / \left(2 M F_0 J_{c\perp}\right)$$

* Applied to CWDC experiment:

$$\frac{e_y}{e_z} = \frac{\gamma \sin \alpha \cos \alpha}{1 + \gamma \cos^2 \alpha} = \frac{(\Gamma^2 - 1) \tan \alpha}{\Gamma^2 + \tan^2 \alpha} \quad \checkmark$$
3. Application
3.1. Approximations to the helical problem

In all cases $I_{tr}(t) = I_0 \sin \omega t$ along each layer

and we obtain $j(z, t)$ across the layers
Model A: influence of the power-law exponent

In this case $\Gamma = 1$
Model A: influence of the anisotropy ratio

In this case $M = 10$ & $\alpha = 67.5^\circ$
Model B: the current flow \((2\alpha = 2\beta = 45^\circ)\)

Anisotropic \(\Rightarrow\) inhomogeneous
4. Conclusions

* Elliptic yield region of current density $J_\perp(J_{\parallel})$
  
  Experimental evidence (CWDC)

  The minimal physical model (unique $F_p$)

* Numerical modelling: the “power-law” $E(J)$
  
  Equivalent $F(J)$ formulation fully tested

  A feasible and sound form of $E(J)$ given

  Next: implementation of $E(J)$ in FE codes . . .
Many thanks for your attention!

http://fmc.unizar.es/people/anabadia/